PERGAMON

International Journal of Solids and Structures 40 (2003) 6999–7013

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Bending and flexure of cylindrically monoclinic elastic cylinders

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Received 17 September 2002

**Abstract**

We consider a circular cylinder of linearly elastic material with cylindrically monoclinic material symmetry. This represents a model for a helically wound composite cable or wire rope. The elastic moduli are allowed to be arbitrary functions of the radius $r$. The cylinder undergoes deformation in which the axis of the cylinder is bent into a plane quartic curve. For the resulting stress field, we obtain exact integrals of the equilibrium equations, and derive simplified expressions for the shear stress resultants and bending moments.


*Keywords:* Elastic cylinders; Cables; Bending; Flexure; Anisotropic elasticity

## 1. Introduction

In a previous paper Crossley et al. (2003) have given exact analytical solutions for bending and flexure of elastic circular cylinders with cylindrical monoclinic symmetry. This theory was designed to model the behaviour of helically wound cables and wire ropes. Deformations that corresponded to simple bending, flexure under end loading, and bending under a uniform load, or self-weight, were analyzed. Among other results, it was shown that because of the handedness of the helical winding, it is necessary to apply bending moments about the $x$-axis as well as about the $y$-axis to produce flexure in the $(x, z)$ plane. Further details, including extensive graphical results, and background information have been given by Crossley (2002).

There is an extensive and well-known literature on bending and flexure of *isotropic* elastic cylinders, that is described in the standard texts, such as Love (1944) or Timoshenko and Goodier (1951). This analysis extends in a very straightforward way to cylindrically orthotropic elastic cylinders provided that the orthotropic axes coincide with the $r$, $\theta$, and $z$ directions of cylindrical polar coordinates $(r, \theta, z)$. This analysis is basic for the formulation of the one-dimensional Saint-Venant theory of bending and flexure of cylinders. Equations governing elastic behaviour of materials with general cylindrical isotropy were formulated by

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0020-7683/$ - see front matter 
doi:10.1016/S0020-7683(03)00351-2

Lekhnitskii (1963) but no explicit solutions were given. Lekhnitskii (1963) also specialised to the case of monoclinic symmetry with the normal cross-sections of the cylinder as planes of elastic reflectional symmetry. However the appropriate continuum model for helically wound cables and ropes requires monoclinic symmetry with the concentric circular cylindrical surfaces $r =$ constant as the surfaces of reflectional symmetry. The relevant governing equations are easily derived (for example Spencer, 1984) but few explicit solutions have been recorded. One such solution by Blouin and Cardou (1989) concerns the problem of extension and torsion of a circular cylinder. There are numerous engineering studies that employ discrete or semi-continuous models of cables and ropes. General references to this work are the book by Costello (1997) and a review by Cardou and Jolicoeur (1997). This paper is concerned only with the continuum theory of linear elasticity.

Thirteen elastic stiffness coefficients $c_{ij}$ are required to characterize the monoclinic symmetry of a linear elastic material. In the model used in Crossley et al. (2003) it was assumed that the cylinder was locally transversely isotropic, with the preferred direction of transverse isotropy coincident with the direction of the helical winding. In this case the 13 coefficients can be expressed in terms of the lay angle $\delta$ and five elastic moduli that characterize transverse isotropy. In order to derive analytical solutions, it was assumed by Crossley et al. (2003) that the transversely isotropic moduli and the lay angle were constants throughout the cylinder (or piecewise constant functions of the radial coordinate $r$ in the case of a layered cylinder), and so consequently that the stiffnesses $c_{ij}$ were constant or piecewise constant.

In general the lay angle $\delta$ is not constant (for example, for a winding of constant pitch $p$ we have $p \tan\delta = 2\pi r$) although it may often be adequately approximated as a piecewise constant function of $r$. If the $c_{ij}$ are functions of $r$ then, for the considered deformations, the governing equations may be reduced to a system of ordinary differential equations, that in general have to be solved numerically. The purpose of this paper is to derive some general results and integrals of the equations that do not require the $c_{ij}$ to be constant or piecewise constant.

The basic elasticity theory is outlined in Section 2. The bending and flexural deformations are described in Section 3 and some integrals of the equilibrium equations are obtained in Section 4. Simplified expressions for the stress resultants and moments are derived in Section 5. Sections 6–8 describe the special cases in which the axis of the cylinder is bent in the $(x, z)$ plane into quadratic, cubic and quartic curves respectively.

## 2. General theory

We consider a circular cylindrical tube of radius $a$ of linearly elastic material. At first all vector and tensor quantities are referred to a system of cylindrical polar coordinates $r$, $\theta$, $z$. In this system the components of the displacement $\mathbf{u}$ are denoted $u_r$, $u_\theta$, $u_z$. The components of the stress tensor and the infinitesimal strain tensor are arranged as the column vectors

$$\begin{aligned} \boldsymbol{\sigma} &= [\sigma_{rr},\ \sigma_{\theta\theta},\ \sigma_{zz},\ \sigma_{\theta z},\ \sigma_{rz},\ \sigma_{r\theta}]^{\mathrm{T}}, \\ \mathbf{e} &= [e_{rr},\ e_{\theta\theta},\ e_{zz},\ 2e_{\theta z},\ 2e_{rz},\ 2e_{r\theta}]^{\mathrm{T}}, \end{aligned} \tag{2.1}$$

respectively, where

$$\begin{aligned} & e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \\ & 2e_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}, \quad 2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \end{aligned} \tag{2.2}$$

The only required material symmetry is reflectional symmetry with respect to the circular cylindrical surfaces $r=$ constant, so the material may be described as cylindrically monoclinic. The corresponding stress–strain relation can be expressed as

$$\boldsymbol{\sigma} = \mathbf{C}\mathbf{e}, \quad \text{where} \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{bmatrix}. \tag{2.3}$$

An important special case arises when the material is locally transversely isotropic, with the axis of transverse isotropy defined by a unit vector $\mathbf{a} = \sin\delta(r)\mathbf{i}_\theta + \cos\delta(r)\mathbf{i}_z$ (where $\mathbf{i}_r, \mathbf{i}_\theta$ and $\mathbf{i}_z$ are unit vectors in the radial, circumferential and axial directions respectively) so that the trajectories of $\mathbf{a}$ are helices lying in the cylindrical surfaces with tangents inclined at the lay angle $\delta(r)$ to the $z$ direction. In this case the elastic stiffnesses $c_{ij}$ $(i,j = 1,2,\ldots,6)$ may be expressed in terms of the angle $\delta$ and five elastic constants. This configuration provides a model for the description of helically wound cables and wires. Further details and references are given in Crossley et al. (2003). For the present analysis there is no extra difficulty in considering the general case in which no restrictions are placed on the $c_{ij}$ (other than those required by positive-definiteness of the strain energy) and so we shall regard the $c_{ij}$ as general functions of the coordinate $r$.

In cylindrical polar coordinates the equations of equilibrium are

$$\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \mathbf{F}\cdot\mathbf{i}_r = 0, \\ &\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \mathbf{F}\cdot\mathbf{i}_\theta = 0, \\ &\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \mathbf{F}\cdot\mathbf{i}_z = 0, \end{aligned} \tag{2.4}$$

where $\mathbf{F}$ is the body force per unit volume.

## 3. Bending and flexural deformations

In addition to the cylindrical polar coordinates $(r,\theta,z)$, we employ rectangular Cartesian coordinates $(x,y,z)$ such that $x = r\cos\theta$, $y = r\sin\theta$, and denote components of $\mathbf{u}$ in the $x$ and $y$ directions by $u_x$, $u_y$ respectively. We seek solutions in which the axis $r=0$ of the cylinder is bent in the $(x,z)$ plane into a parabolic, cubic or quartic curve of the form

$$u_x = \tfrac{1}{2}Cz^2 + \tfrac{1}{6}Dz^3 + \tfrac{1}{24}Ez^4, \tag{3.1}$$

where $C$, $D$ and $E$ are constants. Following the procedure described in Crossley et al. (2003), we consider displacement fields of the form

$$\begin{aligned}
u_r &= \{f_1(r) + zf_3(r) + \tfrac{1}{2}z^2 f_5(r) + \tfrac{1}{2}Cz^2 + \tfrac{1}{6}Dz^3 + \tfrac{1}{24}Ez^4\}\cos\theta \\
&\quad + \{f_2(r) + zf_4(r)\}\sin\theta, \\
u_\theta &= \{g_1(r) + zg_3(r) + \tfrac{1}{2}z^2 g_5(r) - \tfrac{1}{2}Cz^2 - \tfrac{1}{6}Dz^3 - \tfrac{1}{24}Ez^4\}\sin\theta \\
&\quad + \{g_2(r) + zg_4(r)\}\cos\theta, \\
u_z &= \{h_1(r) + zh_3(r) + \tfrac{1}{2}z^2 h_5(r)\}\sin\theta \\
&\quad + \{h_2(r) + zh_4(r) - Crz - \tfrac{1}{2}Drz^2 - \tfrac{1}{6}Erz^3\}\cos\theta.
\end{aligned} \tag{3.2}$$

Hence, from (2.2), the strain tensor $\mathbf{e}$ has the form

$$\mathbf{e} = (\mathbf{e}_{10} + z\mathbf{e}_{11} + \tfrac{1}{2}z^2\mathbf{e}_{12})\cos\theta + (\mathbf{e}_{20} + z\mathbf{e}_{21} + \tfrac{1}{2}z^2\mathbf{e}_{22})\sin\theta, \tag{3.3}$$

where

$$\mathbf{e}_{\alpha\beta} = \left[e_{rr}^{(\alpha\beta)},\ e_{\theta\theta}^{(\alpha\beta)},\ e_{zz}^{(\alpha\beta)},\ 2e_{\theta z}^{(\alpha\beta)},\ 2e_{rz}^{(\alpha\beta)},\ 2e_{r\theta}^{(\alpha\beta)}\right]^{\mathrm{T}}, \quad \alpha = 1,2, \quad \beta = 0,1,2 \tag{3.4}$$

and

$$\begin{aligned}
\mathbf{e}_{10} &= [f_1',\ (f_1 + g_1)/r,\ h_4 - Cr,\ g_4 + h_1/r,\ f_3 + h_2',\ (f_2 - g_2)/r + g_2']^{\mathrm{T}}, \\
\mathbf{e}_{20} &= [f_2',\ (f_2 - g_2)/r,\ h_3,\ g_3 - h_2/r,\ f_4 + h_1',\ -(f_1 + g_1)/r + g_1']^{\mathrm{T}}, \\
\mathbf{e}_{11} &= [f_3',\ (f_3 + g_3)/r,\ -Dr,\ h_3/r,\ f_5 + h_4',\ (f_4 - g_4)/r + g_4']^{\mathrm{T}}, \\
\mathbf{e}_{21} &= [f_4',\ (f_4 - g_4)/r,\ h_5,\ g_5 - h_4/r,\ h_3',\ -(f_3 + g_3)/r + g_3']^{\mathrm{T}}, \\
\mathbf{e}_{12} &= [f_5',\ (f_5 + g_5)/r,\ -Er,\ h_5/r,\ 0,\ 0]^{\mathrm{T}}, \\
\mathbf{e}_{22} &= [0,\ 0,\ 0,\ 0,\ h_5',\ -(f_5 + g_5)/r + g_5']^{\mathrm{T}},
\end{aligned} \tag{3.5}$$

where primes denote derivatives with respect to $r$.

Correspondingly, from (2.3), $\boldsymbol{\sigma}$ has the form

$$\boldsymbol{\sigma} = (\mathbf{t}_1 + z\mathbf{s}_1 + \tfrac{1}{2}z^2\mathbf{p}_1)\cos\theta + (\mathbf{t}_2 + z\mathbf{s}_2 + \tfrac{1}{2}z^2\mathbf{p}_2)\sin\theta, \tag{3.6}$$

where for $\alpha = 1,2$

$$\begin{aligned}
\mathbf{t}_\alpha &= \left[t_{rr}^{(\alpha)},\ t_{\theta\theta}^{(\alpha)},\ t_{zz}^{(\alpha)},\ t_{\theta z}^{(\alpha)},\ t_{rz}^{(\alpha)},\ t_{r\theta}^{(\alpha)}\right]^{\mathrm{T}}, \\
\mathbf{s}_\alpha &= \left[s_{rr}^{(\alpha)},\ s_{\theta\theta}^{(\alpha)},\ s_{zz}^{(\alpha)},\ s_{\theta z}^{(\alpha)},\ s_{rz}^{(\alpha)},\ s_{r\theta}^{(\alpha)}\right]^{\mathrm{T}}, \\
\mathbf{p}_\alpha &= \left[p_{rr}^{(\alpha)},\ p_{\theta\theta}^{(\alpha)},\ p_{zz}^{(\alpha)},\ p_{\theta z}^{(\alpha)},\ p_{rz}^{(\alpha)},\ p_{r\theta}^{(\alpha)}\right]^{\mathrm{T}}.
\end{aligned} \tag{3.7}$$

Hence, from (2.3), the stress–strain relation can be expressed as

$$[\,\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{p}_1 \quad \mathbf{p}_2\,] = \mathbf{C}[\,\mathbf{e}_{10} \quad \mathbf{e}_{20} \quad \mathbf{e}_{11} \quad \mathbf{e}_{21} \quad \mathbf{e}_{12} \quad \mathbf{e}_{22}\,] \tag{3.8}$$

and we note that, in particular

$$p_{rz}^{(1)} = 0, \quad p_{r\theta}^{(1)} = 0, \quad p_{rr}^{(2)} = 0, \quad p_{\theta\theta}^{(2)} = 0, \quad p_{zz}^{(2)} = 0, \quad p_{\theta z}^{(2)} = 0. \tag{3.9}$$

## 4. Integrals of the equilibrium equations

If we now use (3.5), (3.6) and (3.8) to express the cylindrical polar components of $\boldsymbol{\sigma}$ in terms of $f_1, \ldots, f_5, g_1, \ldots, g_5, h_1, \ldots, h_5$, substitute the resulting expressions into the equilibrium equations (2.4), and in each of the equilibrium equations equate the coefficients of $z^p\cos\theta$ and $z^p\sin\theta$ $(p = 0,1,2)$ to zero, we obtain a system of 15 ordinary differential equations for $f_1, \ldots, f_5, g_1, \ldots, g_5, h_1, \ldots, h_5$. This system was formulated by Crossley et al. (2003) for the case in which the elastic stiffnesses $c_{ij}$ are constants, and it was shown there that the system then admits analytical solutions which were developed in detail. The analysis was also extended by Crossley et al. (2003) to the case of a layered cylinder in which the $c_{ij}$ are piecewise constant functions of $r$.

In this section we derive some integrals of the equilibrium equations that do not require the $c_{ij}$ to be constant, but are valid for any (including discontinuous) dependence of the $c_{ij}$ on the radial coordinate $r$. By substituting (3.6) and (3.7) into the equilibrium equations (2.4), and equating to zero the coefficients of $z^0 \cos\theta$ and $z^0 \sin\theta$, there follows

$$\frac{\mathrm{d}t_{rr}^{(1)}}{\mathrm{d}r} + \frac{t_{r\theta}^{(2)}}{r} + s_{rz}^{(1)} + \frac{t_{rr}^{(1)} - t_{\theta\theta}^{(1)}}{r} = -W_0, \tag{4.1}$$

$$\frac{\mathrm{d}t_{rr}^{(2)}}{\mathrm{d}r} - \frac{t_{r\theta}^{(1)}}{r} + s_{rz}^{(2)} + \frac{t_{rr}^{(2)} - t_{\theta\theta}^{(2)}}{r} = 0, \tag{4.2}$$

$$\frac{\mathrm{d}t_{r\theta}^{(1)}}{\mathrm{d}r} + \frac{t_{\theta\theta}^{(2)}}{r} + s_{\theta z}^{(1)} + \frac{2t_{r\theta}^{(1)}}{r} = 0, \tag{4.3}$$

$$\frac{\mathrm{d}t_{r\theta}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta\theta}^{(1)}}{r} + s_{\theta z}^{(2)} + \frac{2t_{r\theta}^{(2)}}{r} = W_0, \tag{4.4}$$

$$\frac{\mathrm{d}t_{rz}^{(1)}}{\mathrm{d}r} + \frac{t_{\theta z}^{(2)}}{r} + s_{zz}^{(1)} + \frac{t_{rz}^{(1)}}{r} = 0, \tag{4.5}$$

$$\frac{\mathrm{d}t_{rz}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta z}^{(1)}}{r} + s_{zz}^{(2)} + \frac{t_{rz}^{(2)}}{r} = 0. \tag{4.6}$$

Here, as in Crossley et al. (2003), the body force $\mathbf{F}$ has been chosen to act in the $x$ direction, so that $\mathbf{F} = W_0(\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta)$.

Similarly, by equating to zero the coefficients of $z\cos\theta$ and $z\sin\theta$ in (2.4), there follows

$$\frac{\mathrm{d}s_{rr}^{(1)}}{\mathrm{d}r} + \frac{s_{r\theta}^{(2)}}{r} + \frac{s_{rr}^{(1)} - s_{\theta\theta}^{(1)}}{r} = 0, \tag{4.7}$$

$$\frac{\mathrm{d}s_{rr}^{(2)}}{\mathrm{d}r} - \frac{s_{r\theta}^{(1)}}{r} + p_{rz}^{(2)} + \frac{s_{rr}^{(2)} - s_{\theta\theta}^{(2)}}{r} = 0, \tag{4.8}$$

$$\frac{\mathrm{d}s_{r\theta}^{(1)}}{\mathrm{d}r} + \frac{s_{\theta\theta}^{(2)}}{r} + p_{\theta z}^{(1)} + \frac{2s_{r\theta}^{(1)}}{r} = 0, \tag{4.9}$$

$$\frac{\mathrm{d}s_{r\theta}^{(2)}}{\mathrm{d}r} - \frac{s_{\theta\theta}^{(1)}}{r} + \frac{2s_{r\theta}^{(2)}}{r} = 0, \tag{4.10}$$

$$\frac{\mathrm{d}s_{rz}^{(1)}}{\mathrm{d}r} + \frac{s_{\theta z}^{(2)}}{r} + p_{zz}^{(1)} + \frac{s_{rz}^{(1)}}{r} = 0, \tag{4.11}$$

$$\frac{\mathrm{d}s_{rz}^{(2)}}{\mathrm{d}r} - \frac{s_{\theta z}^{(1)}}{r} + \frac{s_{rz}^{(2)}}{r} = 0, \tag{4.12}$$

where account has been taken of (3.9).

Further, by equating to zero the coefficients of $z^2\cos\theta$ and $z^2\sin\theta$ in (2.4), and taking account of (3.9), we obtain

$$\frac{dp_{rr}^{(1)}}{dr}+\frac{p_{r\theta}^{(2)}}{r}+\frac{p_{rr}^{(1)}-p_{\theta\theta}^{(1)}}{r}=0, \tag{4.13}$$

$$\frac{dp_{r\theta}^{(2)}}{dr}-\frac{p_{\theta\theta}^{(1)}}{r}+\frac{2p_{r\theta}^{(2)}}{r}=0, \tag{4.14}$$

$$\frac{dp_{rz}^{(2)}}{dr}-\frac{p_{\theta z}^{(1)}}{r}+\frac{p_{rz}^{(2)}}{r}=0. \tag{4.15}$$

By eliminating $p_{\theta\theta}^{(1)}$ from (4.13) and (4.14) there follows

$$\frac{dp_{rr}^{(1)}}{dr}+\frac{p_{rr}^{(1)}}{r}-\frac{dp_{r\theta}^{(2)}}{dr}-\frac{p_{r\theta}^{(2)}}{r}=0$$

and hence

$$\frac{d}{dr}\{r(p_{rr}^{(1)}-p_{r\theta}^{(2)})\}=0$$

and therefore

$$p_{rr}^{(1)}-p_{r\theta}^{(2)}=\frac{G_1}{r}, \tag{4.16}$$

where $G_1$ is constant. Also, from (4.15)

$$p_{\theta z}^{(1)}=\frac{d}{dr}(rp_{rz}^{(2)}). \tag{4.17}$$

Next, from (4.7) and (4.10), by eliminating $s_{\theta\theta}^{(1)}$

$$\frac{ds_{rr}^{(1)}}{dr}+\frac{s_{rr}^{(1)}}{r}-\frac{ds_{r\theta}^{(2)}}{dr}-\frac{s_{r\theta}^{(2)}}{r}=0$$

from which there follows

$$s_{rr}^{(1)}-s_{r\theta}^{(2)}=\frac{B_1}{r}, \tag{4.18}$$

where $B_1$ is constant. Also, from (4.12)

$$s_{\theta z}^{(1)}=\frac{d}{dr}(rs_{rz}^{(2)}). \tag{4.19}$$

Similarly, from (4.8) and (4.9)

$$\frac{ds_{rr}^{(2)}}{dr}+\frac{s_{r\theta}^{(1)}}{r}+p_{rz}^{(2)}+\frac{s_{rr}^{(2)}}{r}+\frac{ds_{r\theta}^{(1)}}{dr}+p_{\theta z}^{(1)}=0 \tag{4.20}$$

and it follows from (4.17) and (4.20) that

$$s_{rr}^{(2)}+s_{r\theta}^{(1)}+rp_{rz}^{(2)}=\frac{B_2}{r}, \tag{4.21}$$

where $B_2$ is another constant. Also, from (4.11)

$$s_{\theta z}^{(2)} + rp_{zz}^{(1)} = -\frac{\mathrm{d}}{\mathrm{d}r}(rs_{rz}^{(1)}). \tag{4.22}$$

By similar arguments, we may obtain from (4.1)–(4.6) the results

$$t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)} = \frac{A_1}{r} - W_0 r - \frac{1}{r}\int_0^r r^2 p_{zz}^{(1)}\,\mathrm{d}r, \tag{4.23}$$

$$t_{rr}^{(2)} + t_{r\theta}^{(1)} + rs_{rz}^{(2)} = \frac{A_2}{r}, \tag{4.24}$$

$$t_{\theta z}^{(1)} - rs_{zz}^{(2)} = \frac{\mathrm{d}}{\mathrm{d}r}(rt_{rz}^{(2)}), \tag{4.25}$$

$$t_{\theta z}^{(2)} + rs_{zz}^{(1)} = -\frac{\mathrm{d}}{\mathrm{d}r}(rt_{rz}^{(1)}). \tag{4.26}$$

If the surface $r = a$ of the cylinder is free from traction, so that at $r = a$

$$\sigma_{rr} = 0, \quad \sigma_{r\theta} = 0, \quad \sigma_{rz} = 0, \tag{4.27}$$

then it follows that the constants $A_2$, $B_1$, $B_2$, and $G_1$ are all zero, and $A_1 = W_0 a^2 + \int_0^a r^2 p_{zz}^{(1)}\,\mathrm{d}r$. Then (4.16), (4.18), (4.21), (4.23) and (4.24) reduce to

$$\begin{aligned} &t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)} = \frac{W_0(a^2 - r^2)}{r} + \frac{1}{r}\int_r^a r^2 p_{zz}^{(1)}\,\mathrm{d}r, \\ &t_{rr}^{(2)} + t_{r\theta}^{(1)} + rs_{rz}^{(2)} = 0, \\ &s_{rr}^{(1)} - s_{r\theta}^{(2)} = 0, \quad s_{rr}^{(2)} + s_{r\theta}^{(1)} + rp_{rz}^{(2)} = 0, \quad p_{rr}^{(1)} - p_{r\theta}^{(2)} = 0. \end{aligned} \tag{4.28}$$

The case of bending and flexure in the $(y, z)$ plane is dealt with by replacing $\theta$ by $\theta + \frac{1}{2}\pi$. In general the deformations in the $(x, z)$ and $(y, z)$ planes are coupled through the boundary conditions, and it is necessary to combine the two sets of solutions. Some examples were described in Crossley et al. (2003).

## 5. Stress resultants and moments

The components of the force on a normal cross-section $z =$ constant of the cylinder are denoted by $(X, Y, Z)$ and the moments of these forces about the $(x, y, z)$ axes by $(M_x, M_y, M_z)$. Thus

$$
\begin{aligned}
X &= \int_0^a \int_0^{2\pi} (\sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta,\\
Y &= \int_0^a \int_0^{2\pi} (\sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta) r\,\mathrm{d}r\,\mathrm{d}\theta,\\
Z &= \int_0^a \int_0^{2\pi} \sigma_{zz} r\,\mathrm{d}r\,\mathrm{d}\theta,\\
M_x &= \int_0^a \int_0^{2\pi} \sigma_{zz}\sin\theta\, r^2\,\mathrm{d}r\,\mathrm{d}\theta,\\
M_y &= -\int_0^a \int_0^{2\pi} \sigma_{zz}\cos\theta\, r^2\,\mathrm{d}r\,\mathrm{d}\theta,\\
M_z &= \int_0^a \int_0^{2\pi} \sigma_{\theta z} r^2\,\mathrm{d}r\,\mathrm{d}\theta.
\end{aligned}
\tag{5.1}
$$

It follows from (3.6) that

$$
\begin{aligned}
X &= \pi\int_0^a \left\{ (t_{rz}^{(1)} - t_{\theta z}^{(2)}) + z(s_{rz}^{(1)} - s_{\theta z}^{(2)}) + \frac{1}{2}z^2(p_{rz}^{(1)} - p_{\theta z}^{(2)}) \right\} r\,\mathrm{d}r,\\
Y &= \pi\int_0^a \left\{ (t_{rz}^{(2)} + t_{\theta z}^{(1)}) + z(s_{rz}^{(2)} + s_{\theta z}^{(1)}) + \frac{1}{2}z^2(p_{rz}^{(2)} + p_{\theta z}^{(1)}) \right\} r\,\mathrm{d}r,\\
Z &= 0,\\
M_x &= \pi\int_0^a \left\{ t_{zz}^{(2)} + zs_{zz}^{(2)} + \frac{1}{2}z^2 p_{zz}^{(2)} \right\} r^2\,\mathrm{d}r,\\
M_y &= -\pi\int_0^a \left\{ t_{zz}^{(1)} + zs_{zz}^{(1)} + \frac{1}{2}z^2 p_{zz}^{(1)} \right\} r^2\,\mathrm{d}r,\\
M_z &= 0.
\end{aligned}
\tag{5.2}
$$

The zero values of $Z$ and $M_z$ reflect the fact that the bending and flexural deformations considered here uncouple from extensional and torsional deformations of the cylindrically monoclinic cylinder.

From (3.9), the expressions for $X$ and $M_x$ simplify to

$$
\begin{aligned}
X &= \pi\int_0^a \{ (t_{rz}^{(1)} - t_{\theta z}^{(2)}) + z(s_{rz}^{(1)} - s_{\theta z}^{(2)}) \} r\,\mathrm{d}r,\\
M_x &= \pi\int_0^a \{ t_{zz}^{(2)} + zs_{zz}^{(2)} \} r^2\,\mathrm{d}r.
\end{aligned}
\tag{5.3}
$$

From (4.22), (4.26) and $(5.3_1)$

$$
\begin{aligned}
X &= \pi\int_0^a \left[ \left\{ (t_{rz}^{(1)} + \frac{\mathrm{d}}{\mathrm{d}r}(rt_{rz}^{(1)}) + rs_{zz}^{(1)} \right\} + z\left\{ (s_{rz}^{(1)} + \frac{\mathrm{d}}{\mathrm{d}r}(rs_{rz}^{(1)}) + rp_{zz}^{(1)} \right\} \right] r\,\mathrm{d}r\\
&= \pi[r^2 t_{rz}^{(1)} + zr^2 s_{rz}^{(1)}]_0^a + \pi\int_0^a s_{zz}^{(1)} r^2\,\mathrm{d}r + \pi z\int_0^a p_{zz}^{(1)} r^2\,\mathrm{d}r.
\end{aligned}
\tag{5.4}
$$

Hence, if the surface $r = a$ is traction-free

$$
X = \pi\int_0^a s_{zz}^{(1)} r^2\,\mathrm{d}r + \pi z\int_0^a p_{zz}^{(1)} r^2\,\mathrm{d}r. \tag{5.5}
$$

However, it follows from (4.28) that

$$\int_0^a p_{zz}^{(1)} r^2 \, \mathrm{d}r = -W_0 a^2 \tag{5.6}$$

(assuming that $t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)}$ is bounded on the axis $r = 0$), and so in this case (5.5) becomes

$$X = \pi \int_0^a s_{zz}^{(1)} r^2 \, \mathrm{d}r - \pi z W_0 a^2. \tag{5.7}$$

Similarly, from (4.17), (4.19), (4.25) and (5.2)

$$\begin{aligned} Y &= \pi \int_0^a \left[ \left\{ t_{rz}^{(2)} + \frac{\mathrm{d}}{\mathrm{d}r}(rt_{rz}^{(2)}) + rs_{zz}^{(2)} \right\} + z \left\{ s_{rz}^{(2)} + \frac{\mathrm{d}}{\mathrm{d}r}(rs_{rz}^{(2)}) \right\} + \frac{1}{2} z^2 \left\{ p_{rz}^{(2)} + \frac{\mathrm{d}}{\mathrm{d}r}(rp_{rz}^{(2)}) \right\} \right] r \, \mathrm{d}r \\ &= \pi \left[ r^2 t_{rz}^{(2)} + zr^2 s_{rz}^{(2)} + \frac{1}{2} z^2 r^2 p_{rz}^{(2)} \right]_0^a + \pi \int_0^a s_{zz}^{(2)} r^2 \, \mathrm{d}r \end{aligned} \tag{5.8}$$

and therefore, when the surface of the cylinder is traction-free

$$Y = \pi \int_0^a s_{zz}^{(2)} r^2 \, \mathrm{d}r. \tag{5.9}$$

It is of interest to note that the resultants and moments (and hence, as described in Crossley et al. (2003), the bending stiffnesses of the cylinder) can be determined with a knowledge only of the stress component $\sigma_{zz}$. We observe also that the stress resultants and moments satisfy the beam equilibrium equations

$$\frac{\mathrm{d}M_x}{\mathrm{d}z} - Y = 0, \quad \frac{\mathrm{d}M_y}{\mathrm{d}z} + X = 0, \tag{5.10}$$

$$\frac{\mathrm{d}X}{\mathrm{d}z} + \pi a^2 W_0 = 0, \quad \frac{\mathrm{d}Y}{\mathrm{d}z} = 0. \tag{5.11}$$

The first of (5.11) confirms that (5.6) holds and hence that the quantity $t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)}$ is bounded on the axis $r = 0$. A proof of this result that is independent of the beam equilibrium equations is given in Section 8.

## 6. Pure bending

The deformation

$$u_r = \{f_1(r) + \tfrac{1}{2}Cz^2\}\cos\theta, \quad u_\theta = \{g_1(r) - \tfrac{1}{2}Cz^2\}\sin\theta, \quad u_z = h_1(r)\sin\theta - Crz\cos\theta \tag{6.1}$$

represents the special case of (3.2) in which the axis of the cylinder is deformed into an arc in the $(x, z)$ plane with constant curvature $C$. It follows that in this case

$$\begin{aligned} &\mathbf{e}_{10} = \left[ f_1', \ (f_1 + g_1)/r, \ -Cr, \ h_1/r, \ 0, \ 0 \right]^{\mathrm{T}}, \\ &\mathbf{e}_{20} = \left[ 0, \ 0, \ 0, \ 0, \ h_1', \ -(f_1 + g_1)/r + g_1' \right]^{\mathrm{T}}, \\ &\mathbf{e}_{11} = \mathbf{0}, \quad \mathbf{e}_{21} = \mathbf{0}, \quad \mathbf{e}_{12} = \mathbf{0}, \quad \mathbf{e}_{22} = \mathbf{0} \end{aligned} \tag{6.2}$$

and so

$$\begin{aligned} &\mathbf{t}_1 = \left[ t_{rr}^{(1)}, \ t_{\theta\theta}^{(1)}, \ t_{zz}^{(1)}, \ t_{\theta z}^{(1)}, \ 0, \ 0 \right]^{\mathrm{T}}, \quad \mathbf{t}_2 = \left[ 0, \ 0, \ 0, \ 0, \ t_{rz}^{(2)}, \ t_{r\theta}^{(2)} \right]^{\mathrm{T}}, \\ &\mathbf{s}_1 = \mathbf{0}, \quad \mathbf{s}_2 = \mathbf{0}, \quad \mathbf{p}_1 = \mathbf{0}, \quad \mathbf{p}_2 = \mathbf{0} \end{aligned} \tag{6.3}$$

and the stress field has the form $\boldsymbol{\sigma} = \mathbf{t}_1 \cos\theta + \mathbf{t}_2 \sin\theta$. The non-trivial equilibrium equations are (4.1), (4.4) and (4.6) which, with zero body force, reduce to

$$\frac{\mathrm{d}t_{rr}^{(1)}}{\mathrm{d}r} + \frac{t_{r\theta}^{(2)}}{r} + \frac{t_{rr}^{(1)} - t_{\theta\theta}^{(1)}}{r} = 0, \quad \frac{\mathrm{d}t_{r\theta}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta\theta}^{(1)}}{r} + \frac{2t_{r\theta}^{(2)}}{r} = 0, \quad \frac{\mathrm{d}t_{rz}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta z}^{(1)}}{r} + \frac{t_{rz}^{(2)}}{r} = 0. \tag{6.4}$$

These determine $f_1$, $g_1$ and $h_1$ in terms of $C$. If the surface of the cylinder is traction-free, so that the boundary conditions are homogeneous, then the solution of (6.4) has the form

$$(f_1(r), g_1(r), h_1(r)) = C(\bar{f}(r), \bar{g}(r), \bar{h}(r)), \tag{6.5}$$

where $\bar{f}(r)$, $\bar{g}(r)$, $\bar{h}(r)$ are the solutions of (6.4) with $C = 1$. The stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are then given by (3.8) and (6.2). Also when the surface is traction-free

$$t_{r\theta}^{(2)} = t_{rr}^{(1)}, \quad rt_{rz}^{(2)} = -\int_r^a t_{\theta z}^{(1)}\,\mathrm{d}r \tag{6.6}$$

and it follows from (6.6) that if $t_{r\theta}^{(2)} = 0$ on any surface (for example at a frictionless interface between layers in a composite cylinder) then also $t_{rr}^{(1)} = 0$ on that surface.

This deformation in which the cylinder axis deforms to an arc in the $(x, z)$ plane is maintained by the constant bending moment

$$M_y = -\pi\int_0^a t_{zz}^{(1)} r^2\,\mathrm{d}r = -\pi C\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2\,\mathrm{d}r \tag{6.7}$$

and the bending rigidity $b_s$ is given by $M_y = Cb_s$, so that

$$b_s = -\pi\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2\,\mathrm{d}r. \tag{6.8}$$

The other bending moment $M_x$ and the shear forces are zero.

## 7. Flexure with a uniform shear force

In the deformation

$$\begin{aligned} u_r &= \left\{zf_3(r) + \frac{1}{6}Dz^3\right\}\cos\theta + f_2(r)\sin\theta, \\ u_\theta &= \left\{zg_3(r) - \frac{1}{6}Dz^3\right\}\sin\theta + g_2(r)\cos\theta, \\ u_z &= zh_3(r)\sin\theta + \left\{h_2(r) - \frac{1}{2}Drz^2\right\}\cos\theta, \end{aligned} \tag{7.1}$$

the axis $r = 0$ of the cylinder is deformed into a plane cubic curve. In this case

$$\begin{aligned} \mathbf{e}_{10} &= [0,\ 0,\ 0,\ 0,\ f_3 + h_2',\ (f_2 - g_2)/r + g_2']^{\mathrm{T}}, \\ \mathbf{e}_{20} &= [f_2',\ (f_2 - g_2)/r,\ h_3,\ g_3 - h_2/r,\ 0,\ 0]^{\mathrm{T}}, \\ \mathbf{e}_{11} &= [f_3',\ (f_3 + g_3)/r,\ -Dr,\ h_3/r,\ 0,\ 0]^{\mathrm{T}}, \\ \mathbf{e}_{21} &= [0,\ 0,\ 0,\ 0,\ h_3',\ -(f_3 + g_3)/r + g_3']^{\mathrm{T}}, \\ \mathbf{e}_{12} &= \mathbf{0}, \quad \mathbf{e}_{22} = \mathbf{0} \end{aligned} \tag{7.2}$$

and therefore

$$\begin{aligned}
&\mathbf{t}_1 = \left[0,\ 0,\ 0,\ 0,\ t_{rz}^{(1)},\ t_{r\theta}^{(1)}\right]^{\mathrm{T}}, \quad \mathbf{t}_2 = \left[t_{rr}^{(2)},\ t_{\theta\theta}^{(2)},\ t_{zz}^{(2)},\ t_{\theta z}^{(2)},\ 0,\ 0\right]^{\mathrm{T}},\\
&\mathbf{s}_1 = \left[s_{rr}^{(1)},\ s_{\theta\theta}^{(1)},\ s_{zz}^{(1)},\ s_{\theta z}^{(1)},\ 0,\ 0\right]^{\mathrm{T}}, \quad \mathbf{s}_2 = \left[0,\ 0,\ 0,\ 0,\ s_{rz}^{(2)},\ s_{r\theta}^{(2)}\right]^{\mathrm{T}},\\
&\mathbf{p}_1 = \mathbf{0}, \quad \mathbf{p}_2 = \mathbf{0}
\end{aligned} \tag{7.3}$$

and so the stress field has the form $\boldsymbol{\sigma} = (\mathbf{t}_1 + z\mathbf{s}_1)\cos\theta + (\mathbf{t}_2 + z\mathbf{s}_2)\sin\theta$. The non-trivial equilibrium equations are (4.2), (4.3), (4.5), (4.7), (4.10) and (4.12), which determine $f_2, g_2, h_2, f_3, g_3$ and $h_3$ in terms of $D$. Inspection shows that (4.7), (4.10) and (4.12) are identical to (6.4), except that $(f_1, g_1, h_1, C)$ are replaced by $(f_3, g_3, h_3.D)$. Hence if the cylinder surface is free of traction, (4.7), (4.10) and (4.12) have the solution

$$(f_3(r), g_3(r), h_3(r)) = D(\bar{f}(r), \bar{g}(r), \bar{h}(r)), \tag{7.4}$$

and the stress vectors $\mathbf{s}_1$and $\mathbf{s}_2$ are then given by (3.8) and (7.2) and are identical to the stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ in the bending problem when $C$ is replaced by $D$. Equations (4.2), (4.3) and (4.5) then take the forms

$$\begin{aligned}
&\frac{\mathrm{d}t_{rr}^{(2)}}{\mathrm{d}r} - \frac{t_{r\theta}^{(1)}}{r} + \frac{t_{rr}^{(2)} - t_{\theta\theta}^{(2)}}{r} = -s_{rz}^{(2)},\\
&\frac{\mathrm{d}t_{r\theta}^{(1)}}{\mathrm{d}r} + \frac{t_{\theta\theta}^{(2)}}{r} + \frac{2t_{r\theta}^{(1)}}{r} = -s_{\theta z}^{(1)},\\
&\frac{\mathrm{d}t_{rz}^{(1)}}{\mathrm{d}r} + \frac{t_{\theta z}^{(2)}}{r} + \frac{t_{rz}^{(1)}}{r} = -s_{zz}^{(1)}
\end{aligned} \tag{7.5}$$

which, since the right-hand sides are now known, and when the surface is traction-free, determine the solution for $f_2$, $g_2$ and $h_2$ to be of the form

$$(f_2(r), g_2(r), h_2(r)) = D(\hat{f}(r), \hat{g}(r), \hat{h}(r)), \tag{7.6}$$

where $(\hat{f}(r), \hat{g}(r), \hat{h}(r))$ is the solution of (7.5) with $D = 1$. When $(f_2, g_2, h_2)$ are determined, the stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are given by (3.8) and (7.2).

If the cylinder surface is traction-free then $t_{rr}^{(2)} + t_{r\theta}^{(1)} + rs_{rz}^{(2)} = 0$ and $s_{rr}^{(1)} - s_{r\theta}^{(2)} = 0$. As in the case of pure bending, if $\sigma_{r\theta} = 0$ and $\sigma_{rz} = 0$ on any cylindrical surface $r =$ constant, then also $\sigma_{rr} = 0$ on that surface. Thus in a composite layered cylinder with smooth frictionless contact between the layers, there is zero normal pressure at the interfaces between the layers. This was observed in numerical calculations by Jolicoeur and Cardou (1994) for the pure bending problem and has also been confirmed numerically by Crossley (2002) for bending and flexure.

The shear stress resultants and bending moments are

$$\begin{aligned}
&X = \pi\int_0^a s_{zz}^{(1)} r^2\,\mathrm{d}r = \pi D\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2\,\mathrm{d}r, \quad Y = 0,\\
&M_x = \pi\int_0^a t_{zz}^{(2)} r^2\,\mathrm{d}r = \pi D\int_0^a \{c_{13}\hat{f}' + c_{23}(\hat{f} + \hat{g})/r - rc_{33} + c_{34}\hat{h}/r\} r^2\,\mathrm{d}r,\\
&M_y = -\pi z\int_0^a s_{zz}^{(1)} r^2\,\mathrm{d}r = -\pi Dz\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2\,\mathrm{d}r.
\end{aligned} \tag{7.7}$$

Hence in order to maintain the deformation in which the axis of the cylinder remains in the $(x, z)$ plane, it is necessary to apply a bending moment $M_x$ about the $x$-axis in addition to the stress resultant $X$ and the bending moment $M_y$. It may be observed from the pure bending solution (6.7) and (6.8) that $X = -Db_s$ and that $M_y = Dzb_s$.

Crossley et al. (2003) formulated a one-dimensional model for elastic beams with curvilinear monoclinic symmetry, in which the constitutive equations for the shear forces and bending moments are

$$b_s \frac{d^2 u_x}{dz^2} = M_y(z) - \alpha Y(z),$$
$$b_s \frac{d^2 u_y}{dz^2} = -M_x(z) + \alpha X(z), \tag{7.8}$$

where $b_s$ is the bending stiffness of the cylinder and $\alpha$ is a material constant that characterizes the coupling between the bending moment and the shear force in the direction normal to the plane of bending. By noting that in the deformation (6.1) $d^2u_x/dz^2 = C$ and $d^2u_y/dz^2 = 0$, and in the deformation (7.1) $d^2u_x/dz^2 = Dz$ and $d^2u_y/dz^2 = 0$, it follows from (6.7) and (7.7) that we can make the identifications

$$b_s = -\pi \int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2 \, dr,$$
$$\alpha b_s = -\pi \int_0^a \{c_{13}\hat{f}' + c_{23}(\hat{f} + \hat{g})/r - rc_{33} + c_{34}\hat{h}/r\} r^2 \, dr. \tag{7.9}$$

## 8. Bending under uniform load

Finally, we examine the case in which $C = 0$, $D = 0$, with $E \neq 0$ and $W_0 \neq 0$, so that the axis of the cylinder is bent into a quartic curve in the $(x, z)$ plane. The displacement becomes

$$u_r = \left\{ f_1(r) + \frac{1}{2}z^2 f_5(r) + \frac{1}{24}Ez^4 \right\} \cos\theta + zf_4(r)\sin\theta,$$
$$u_\theta = \left\{ g_1(r) + \frac{1}{2}z^2 g_5(r) - \frac{1}{24}Ez^4 \right\} \sin\theta + zg_4(r)\cos\theta, \tag{8.1}$$
$$u_z = \left\{ h_1(r) + \frac{1}{2}z^2 h_5(r) \right\} \sin\theta + \left\{ zh_4(r) - \frac{1}{6}Erz^3 \right\} \cos\theta.$$

The corresponding strain is

$$\mathbf{e}_{10} = [f_1', \ (f_1 + g_1)/r, \ h_4, \ g_4 + h_1/r, \ 0, \ 0]^T,$$
$$\mathbf{e}_{20} = [0, \ 0, \ 0, \ 0, \ f_4 + h_1', \ -(f_1 + g_1)/r + g_1']^T,$$
$$\mathbf{e}_{11} = [0, \ 0, \ 0, \ 0, \ f_5 + h_4', \ (f_4 - g_4)/r + g_4']^T,$$
$$\mathbf{e}_{21} = [f_4', \ (f_4 - g_4)/r, \ h_5, \ g_5 - h_4/r, \ 0, \ 0]^T, \tag{8.2}$$
$$\mathbf{e}_{12} = [f_5', \ (f_5 + g_5)/r, \ -Er, \ h_5/r, \ 0, \ 0]^T,$$
$$\mathbf{e}_{22} = [0, \ 0, \ 0, \ 0, \ h_5', \ -(f_5 + g_5)/r + g_5']^T$$

and therefore the stress field is of the form

$$\boldsymbol{\sigma} = \left( \mathbf{t}_1 + z\mathbf{s}_1 + \frac{1}{2}z^2\mathbf{p}_1 \right) \cos\theta + \left( \mathbf{t}_2 + z\mathbf{s}_2 + \frac{1}{2}z^2\mathbf{p}_2 \right) \sin\theta,$$

where

$$\mathbf{t}_1 = \left[ t_{rr}^{(1)}, \ t_{\theta\theta}^{(1)}, \ t_{zz}^{(1)}, \ t_{\theta z}^{(1)}, \ 0, \ 0 \right]^T, \quad \mathbf{t}_2 = \left[ 0, \ 0, \ 0, \ 0, \ t_{rz}^{(2)}, \ t_{r\theta}^{(2)} \right]^T,$$
$$\mathbf{s}_1 = \left[ 0, \ 0, \ 0, \ 0, \ s_{rz}^{(1)}, \ s_{r\theta}^{(1)} \right]^T, \quad \mathbf{s}_2 = \left[ s_{rr}^{(2)}, \ s_{\theta\theta}^{(2)}, \ s_{zz}^{(2)}, \ s_{\theta z}^{(2)}, \ 0, \ 0 \right]^T, \tag{8.3}$$
$$\mathbf{p}_1 = \left[ p_{rr}^{(1)}, \ p_{\theta\theta}^{(1)}, \ p_{zz}^{(1)}, \ p_{\theta z}^{(1)}, \ 0, \ 0 \right]^T, \quad \mathbf{p}_2 = \left[ 0, \ 0, \ 0, \ 0, \ p_{rz}^{(2)}, \ p_{r\theta}^{(2)} \right]^T.$$

The non-trivial equilibrium equations are (4.1), (4.4), (4.6), (4.8), (4.9), (4.11) and (4.13)–(4.15) which determine $f_1, g_1, h_1, f_4, g_4, h_4, f_5, g_5, h_5$ in terms of $E$ and $W_0$. By inspection it can be seen that (4.13)–(4.15) are identical in form to (6.4), and (4.8), (4.9) and (4.11) are identical in form to (7.5), if $(f_1, g_1, h_1, f_2, g_2, h_2, D)$ are replaced by $(f_4, g_4, h_4, f_5, g_5, h_5, E)$. Hence, if the cylinder surface is traction-free, there follow immediately

$$\begin{aligned}(f_5(r), g_5(r), h_5(r)) &= E(\bar{f}(r), \bar{g}(r), \bar{h}(r)),\\ (f_4(r), g_4(r), h_4(r)) &= E(\hat{f}(r), \hat{g}(r), \hat{h}(r)).\end{aligned} \tag{8.4}$$

Hence $\mathbf{e}_{11}$, $\mathbf{e}_{12}$, $\mathbf{e}_{21}$ and $\mathbf{e}_{22}$ are determined, and therefore $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{p}_1$ and $\mathbf{p}_2$ are known, In particular, we have

$$p_{zz}^{(1)} = E\{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} \tag{8.5}$$

and therefore, from (5.6), it follows that $E$ and $W_0$ are not independent, but are related as

$$E\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\}r^2\,\mathrm{d}r = -W_0 a^2, \tag{8.6}$$

or, from (7.9)

$$E = \pi a^2 W_0/b_s. \tag{8.7}$$

Eqs. (4.1), (4.4) and (4.6) can then be written as

$$\begin{aligned}&\frac{\mathrm{d}t_{rr}^{(1)}}{\mathrm{d}r} + \frac{t_{r\theta}^{(2)}}{r} + \frac{t_{rr}^{(1)} - t_{\theta\theta}^{(1)}}{r} = -W_0 - s_{rz}^{(1)},\\ &\frac{\mathrm{d}t_{r\theta}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta\theta}^{(1)}}{r} + \frac{2t_{r\theta}^{(2)}}{r} = W_0 - s_{\theta z}^{(2)},\\ &\frac{\mathrm{d}t_{rz}^{(2)}}{\mathrm{d}r} - \frac{t_{\theta z}^{(1)}}{r} + \frac{t_{rz}^{(2)}}{r} = -s_{zz}^{(2)}.\end{aligned} \tag{8.8}$$

If the cylinder surface is traction-free then

$$\begin{aligned}&t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)} = \frac{W_0(a^2 - r^2)}{r} + \frac{1}{r}\int_r^a p_{zz}^{(1)} r^2\,\mathrm{d}r,\\ &s_{rr}^{(2)} + s_{r\theta}^{(1)} + rp_{rz}^{(2)} = 0, \quad p_{rr}^{(1)} - p_{r\theta}^{(2)} = 0.\end{aligned} \tag{8.9}$$

The shear stress resultants and bending moments are

$$\begin{aligned}X &= \pi z\int_0^a p_{zz}^{(1)} r^2\,\mathrm{d}r = \pi Ez\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\}r^2\,\mathrm{d}r = -Ezb_s,\\ Y &= \pi\int_0^a s_{zz}^{(2)} r^2\,\mathrm{d}r = \pi E\int_0^a \{c_{13}\hat{f}' + c_{23}(\hat{f} + \hat{g})/r - rc_{33} + c_{34}\hat{h}/r\}r^2\,\mathrm{d}r = -E\alpha b_s,\\ M_x &= \pi z\int_0^a s_{zz}^{(2)} r^2\,\mathrm{d}r = \pi Ez\int_0^a \{c_{13}\hat{f}' + c_{23}(\hat{f} + \hat{g})/r - rc_{33} + c_{34}\hat{h}/r\}r^2\,\mathrm{d}r = -Ez\alpha b_s,\\ M_y &= -\pi\int_0^a t_{zz}^{(1)} r^2\,\mathrm{d}r - \frac{1}{2}\pi z^2\int_0^a p_{zz}^{(1)} r^2\,\mathrm{d}r\\ &= M_0 - \frac{1}{2}\pi Ez^2\int_0^a \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\}r^2\,\mathrm{d}r = M_0 + \frac{1}{2}Ez^2 b_s,\end{aligned} \tag{8.10}$$

where

$$M_0 = -\pi\int_0^a t_{zz}^{(1)} r^2\,\mathrm{d}r. \tag{8.11}$$

Therefore in order to maintain the deformation of the cylinder axis into a quartic curve in the $(x, z)$ plane it is necessary to apply shear forces in both $x$ and $y$ directions and bending moments about both $x$ and $y$ axes. By taking into account (8.7) it is easily seen that the beam equilibrium equations (5.10) and (5.11) are satisfied.

When $f_4$, $g_4$, $h_4$, $f_5$, $g_5$, $h_5$ are determined it then follows from (8.8) that $f_1$, $g_1$ and $h_1$ are of the form

$$(f_1(r), g_1(r), h_1(r)) = E(\tilde{f}(r), \tilde{g}(r), \tilde{h}(r)), \tag{8.12}$$

where $\tilde{f}(r)$, $\tilde{g}(r)$, $\tilde{h}(r)$ are the solution of (8.8) for $f_1(r)$, $g_1(r)$, $h_1(r)$ in the case $E = 1$ (and therefore $W_0 = b_s/\pi a^2$). With (8.7), the first of (8.9) becomes

$$t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)} = \frac{Eb_s(a^2 - r^2)}{\pi a^2 r} + \frac{1}{r}\int_r^a p_{zz}^{(1)} r^2 \, \mathrm{d}r. \tag{8.13}$$

Hence, from (7.9) and (8.5)

$$\begin{aligned} t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)} &= -\frac{(a^2 - r^2)}{a^2 r}\int_0^a p_{zz}^{(1)} r^2 \, \mathrm{d}r + \frac{1}{r}\int_r^a p_{zz}^{(1)} r^2 \, \mathrm{d}r. \\ &= \frac{r}{a^2}\int_0^a p_{zz}^{(1)} r^2 \, \mathrm{d}r - \frac{1}{r}\int_0^r p_{zz}^{(1)} r^2 \, \mathrm{d}r \\ &= -E\left[\frac{rb_s}{\pi a^2} + \frac{1}{r}\int_0^r \{c_{13}\bar{f}' + c_{23}(\bar{f} + \bar{g})/r - rc_{33} + c_{34}\bar{h}/r\} r^2 \, \mathrm{d}r\right]. \end{aligned} \tag{8.14}$$

This gives independent confirmation that $t_{rr}^{(1)} - t_{r\theta}^{(2)} + rs_{rz}^{(1)}$ is finite on the axis $r = 0$, as was assumed in (5.6) and, implicitly, in (8.6). Further, at any frictionless circular cylindrical surface $r = r_0$ at which $\sigma_{r\theta} = 0$ and $\sigma_{rz} = 0$, the radial stress component $\sigma_{rr}$ is independent of $z$ and has the form $\sigma_{rr} = t_{rr}^{(1)} \cos\theta$.

The solutions described in Sections 6–8, together with the corresponding solutions for bending in the $(y, z)$ plane and rigid body displacements, may be combined in various ways to give solutions to boundary value problems of interest. For example, Crossley et al. (2003) analyzed the cantilever problem and the catenary problem for a helically reinforced cylinder with constant $c_{ij}$.

## Acknowledgements

The work of JAC was supported by an EPSRC CASE Research Studentship with British Energy plc as the cooperating body. The support of EPSRC and British Energy is gratefully acknowledged.

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